Parameter Uncertainty Computation in Static Linear Models

Stéphane Ploix – Olivier Adrot – José Ragot

Centre de Recherche en Automatique de Nancy – CNRS UPRES-A n°7039
Institut National Polytechnique de Lorraine
2, avenue de la forêt de Haye – 54516 Vandoeuvre Cedex, France
Phone: +33 3 83 59 59 59 – Fax: +33 3 83 59 56 44
E-mail: {sploix, oadrot, jragot}@ensem.u-nancy.fr

Abstract

Static linear models characterized by bounded uncertainties on both the equation error and the parameters are studied. The additive equation error is assumed to belong to an interval while the parameters fluctuate inside a time-invariant bounded domain. An algorithm is proposed for evaluating different bounded domains. The algorithm can be extended to cope with the determination of the central value of the domain containing the parameters. Contrary to most of traditional estimators, the resulting estimator takes into account the distribution of the uncertainties of a model.

Key words: *uncertain linear models, bounding approach, estimation theory*

1. Introduction

The application of the bounding approach to parameter estimation started in the eighties, especially with the works of Fogel et Huang [3] who used ellipsoidal domains, to be followed by those of Milanese et Belforte [5] who worked with orthotopes. Walter and Piet-Lahanier [12] on the one hand, and Mo and Norton [7] on the other, used domains in the form of polytopes. The main results are presented in the book published by Milanese et al [6]. To summarize, the problem of parameter estimation amounts to the determination of the set of parameter values (Feasible Parameter Set: FPS*y*) each of which explains all the available observations whenever the equation error is bounded. These techniques were originally designed to deal with linear models. They have since been extended by Jaulin and Walter [4] to cope with nonlinear models as well. Despite the resemblance, the problem we are considering is different. Indeed, we are not so much concerned about fact the values of the parameters of a model are determined, but the fact these values belong to a domain characterized solely by the invariance of its form. After the formalization of the problem, we will dwell at length on the method for finding the characteristics of this domain, the underlying consideration being that to the best solution must correspond to the most precise model.

The knowledge of uncertainties may be crucial in fault diagnosis [11] or in control system [1].

2. Problem formulation

By analogy with stochastic variables, we introduce the concept of abstract space, denoted by $\mathcal{A}(\ )$, so as to facilitate the application of bounding approach to the investigation of model's uncertainties. If X represents a bounded variable, that is, if all that is known about it is the space to which this variable belongs, then we will denote that space by $\mathcal{A}(X)$. The symbol x will then refer to a particular realization of X. But for the sake of simplification, we will allow the notion of the realization of x merge with that of the bounded variable X itself. In other words, x will thus at times designate the bounded variable X and other times its realization. In the latter case, $\mathcal{A}(x)$ will then designate the abstract space of the bounded variable.

We are interested in uncertain linear models with respect to parameters and observations, those topped with tilde symbol, '~'. By the term uncertain, we mean that we are dealing with parameters, some of which are bounded variables. Let us denote by $\{\tilde{y}_1,\cdots,\tilde{y}_h\}$, the set of real numbers reassembling the values obtained through observations and by $\{\tilde{x}_1,\cdots,\tilde{x}_h\}$ the set of vectors in $\mathbb{R}^p$ verifying the following relations: $\forall k\in\{1,\ldots,h\}$, $\tilde{y}_k=\tilde{x}_k^{\mathrm{T}}\theta_k+\varepsilon_k$. The term ε_k corresponds to a bounded centered variable defined by $\mathcal{A}(\varepsilon_k)=[-\delta_\varepsilon,\delta_\varepsilon]$ ($\delta_\varepsilon\in\mathbb{R}^+$). The vectors θ_k also represent mutually independent bounded variables with an invariant abstract space $\Theta=\mathcal{A}(\theta_k)$. It will be assumed that this space can be described with a parallelotope that is centered on the value $\theta_c\in\mathbb{R}^p$: $\theta_k=\theta_c+T(\lambda)\upsilon_k$ where υ_k is a vector in $\mathbb{R}^q$ the components of which are normalized bounded variables and the matrix $T(\lambda)$ will be further detailed. Moreover the vector υ_k is supposed to be independent of ε_k. In other words, the abstract space $\mathcal{A}(\upsilon_k)$ corresponds to a unit ball generated by an infinite norm ($\mathcal{A}(\upsilon_k)\equiv\{\vartheta\in\mathbb{R}^q/\|\vartheta\|_\infty\le 1\}$). λ is a vector the q positive components of which characterize the abstract space Θ and assumed to be determined**.**

To determine the characteristics of an uncertain model amounts to finding a $(p+q+1)$-tuple of coefficients $(\theta_c,\lambda,\delta_\varepsilon)$ such that the model can explain the most precisely the associated set of observations:

$$\forall k \in \{1,\ldots,h\}, \tilde{y}_k \in \mathcal{A}\left(\tilde{x}_k^{\mathrm{T}}\theta_c + \tilde{x}_k^{\mathrm{T}} T(\lambda)\upsilon_k + \varepsilon_k\right) \quad (1)$$

In any case, it is not every model satisfying (1) that interests us. We are interested only in the most precise of such models. It is therefore necessary to specify the defining criterion of the precision. Let us evaluate the abstract space appearing in relation (1) using the results of interval arithmetic ([7] and [10]):

$$\mathcal{A}\left(\tilde{x}_k^{\mathrm{T}}\theta_c + \tilde{x}_k^{\mathrm{T}} T(\lambda)\upsilon_k + \varepsilon_k\right) = \quad (2)$$
$$\left[\tilde{x}_k^{\mathrm{T}}\theta_c - \left\|\tilde{x}_k^{\mathrm{T}} T(\lambda)\right\|_1 - \delta_\varepsilon , \tilde{x}_k^{\mathrm{T}}\theta_c + \left\|\tilde{x}_k^{\mathrm{T}} T(\lambda)\right\|_1 + \delta_\varepsilon\right]$$

The width of the interval obtained is equal to $j_k = 2\left\|\tilde{x}_k^{\mathrm{T}} T(\lambda)\right\|_1 + 2\delta_\varepsilon$. The culmination of these widths on the horizon h considered provides the model's precision criterion:

$$J = 2h\delta_\varepsilon + 2\sum_{k=1}^{h}\left\|\tilde{x}_k^{\mathrm{T}} T(\lambda)\right\|_1$$

But choosing matrix $T(\lambda)$ in an arbitrary manner will make it difficult to anticipate the influence of a variation in the parameter λ on the precision. To get round this difficulty, we will limit ourselves to matrices having one of the following structures:

$$T(\lambda) = \lambda T_0 \;:\; \lambda \in \mathbb{R}^{*+} \quad (3a)$$
$$T(\lambda) = \left[\lambda_1 t_1 \quad \ldots \quad \lambda_q t_q\right] \;:\; \lambda_i \in \mathbb{R}^{*+} \quad (3b)$$

where t_i is a vector in $\mathbb{R}^p$. Structure (3a) corresponds to a predetermined shape of the abstract space $\mathcal{A}(\theta_k)$: only the size of this domain remains unknown. By contrast, structure (3b) does not specify the shape of the domain $\mathcal{A}(\theta_k)$: it depends of λ. This structure corresponds to cases where the individual influences of the uncertainties υ_i all intervene in the same direction t_i.

The precision criteria for (3a) and (3b) are respectively:

$$J(\delta_\varepsilon,\lambda,) = 2h\delta_\varepsilon + 2\lambda\sum_{k=1}^{h}\left\|\tilde{x}_k^{\mathrm{T}} T_0\right\|_1 \quad (4a)$$
$$J(\delta_\varepsilon,\lambda) = 2h\delta_\varepsilon + 2\chi^{\mathrm{T}}\lambda \quad (4b)$$

with $\lambda=[\lambda_1\ldots\lambda_q]^{\mathrm{T}}$, $\chi_k^{\mathrm{T}}=[|\tilde{x}_k^{\mathrm{T}} t_1|\ldots|\tilde{x}_k^{\mathrm{T}} t_q|]$ and $\chi = \sum_{k=1}^{h}\chi_k$.

The problem then comes to finding the characteristics θ_c, δ_ε and λ such that relations (1) are satisfied and which minimize the corresponding criteria (4). Notice that the central parameter vector, θ_c, is invariant. If θ_c is not known a priori, it can be obtained using a classical estimator and minimizing an α-norm $\|\;\|_\alpha$ of the equation error ε_k raised to the power β:

$$\theta_c = arg\left(\min_{\theta_c}\left(\sum_{k=1}^{h}\left\|\tilde{y}_k - \tilde{x}_k^{\mathrm{T}}\theta_c\right\|_\alpha^\beta\right)\right)$$

or using the *most precise estimator* (to which we will return later in the sequel) which minimizes criteria (4).

The problem can then be summarized as one of computing the characteristics δ_ε and λ at a θ_c given (θ_c and δ_ε will be adjusted by iteration in the case of the *most precise estimator*).

3. Solving for pre-determined shape

Let us first consider the special case of structure (3a). We deduce from relations (1) and (2) that the constraints to verify can also be written as follows: $\forall k\in\{1,\ldots,h\}$,

$$-\lambda\left\|\tilde{x}_k^{\mathrm{T}} T_0\right\|_1 - \delta_\varepsilon \le \tilde{y}_k - \tilde{x}_k^{\mathrm{T}}\theta_c \le \lambda\left\|\tilde{x}_k^{\mathrm{T}} T_0\right\|_1 + \delta_\varepsilon$$

or still as: $\forall k\in\{1,\ldots,h\}$,

$$\lambda \ge max\left(\frac{\tilde{y}_k - \tilde{x}_k^{\mathrm{T}}\theta_c - \delta_\varepsilon}{\left\|x_k^{\mathrm{T}} T_0\right\|}, \frac{\tilde{x}_k^{\mathrm{T}}\theta_c - \tilde{y}_k - \delta_\varepsilon}{\left\|x_k^{\mathrm{T}} T_0\right\|}\right) \quad (5)$$

In view of the precision criterion (4a), it is clear that the smaller the positive characteristic λ, the better the precision. Thus, taking (4a) into account, we deduce from (5) that the best value of λ associated with a fixed δ_ε, in the sense of the criterion, is given by:

$$\lambda = \sup_{k\in\{1,\cdots,h\}}\left(max\left(0,\frac{\tilde{y}_k - \tilde{x}_k^{\mathrm{T}}\theta_c - \delta_\varepsilon}{\left\|x_k^{\mathrm{T}} T_0\right\|_1}, \frac{\tilde{x}_k^{\mathrm{T}}\theta_c - \tilde{y}_k - \delta_\varepsilon}{\left\|x_k^{\mathrm{T}} T_0\right\|_1}\right)\right)$$

It is possible to have an a priori knowledge of the characteristic value δ_ε. Otherwise, one can evaluate the different values of $(\theta_c,\ \delta_\varepsilon,\lambda(\delta_\varepsilon))$ and keep the one that minimizes the criteria (4) (cf. figure V of the illustration).

4. Solving for undetermined shape

It can be deduced from (1) and (2) that when the matrix $T(\lambda)$ has the shape given in (3b), the following constraint must be verified: $\forall k\in\{1,\ldots,h\}$,

$$\tilde{x}_k^{\mathrm{T}}\theta_c - \chi_k^{\mathrm{T}}\lambda - \delta_\varepsilon \le \tilde{y}_k \le \tilde{x}_k^{\mathrm{T}}\theta_c + \chi_k^{\mathrm{T}}\lambda + \delta_\varepsilon$$

or: $\forall k\in\{1,\ldots,h\}$,

$$\chi_k^{\mathrm{T}}\lambda \ge max\left(\tilde{y}_k - \tilde{x}_k^{\mathrm{T}}\theta_c - \delta_\varepsilon, \tilde{x}_k^{\mathrm{T}}\theta_c - \tilde{y}_k - \delta_\varepsilon\right) \quad (7)$$

In this case, the problem becomes one of minimizing criterion (4b) while respecting the constraints imposed by (7) instead of (1) and (2). Each of the constraints in (7) defines a $\mathbb{R}^q$ half-space the frontier of which is determined by a $\mathbb{R}^{q-1}$ plane domain. The intersection of the different half-spaces determines a $\mathbb{R}^q$ forbidden space for λ (cf. figure I).

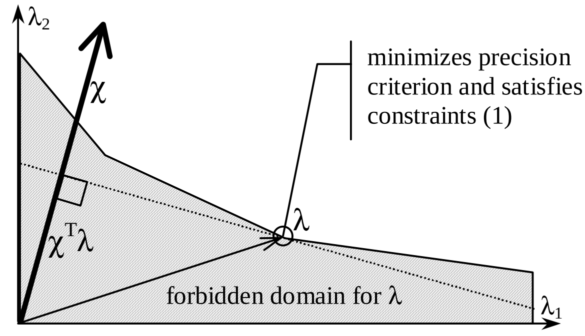

figure I – Graphical solution of the constrained minimization problem

The most precise solution corresponds to the admissible value of λ that minimizes the scalar product $\chi^{T}\lambda$ in (4b). The admissible domain being convex (the intersection of half-planes is necessarily convex), one of the polytope vertices of the admissible domain necessarily corresponds to the smallest orthogonal projection, on the straight line determined by χ, of an admissible value of λ.

The polytope calculation is a well-known technique. Mo and Norton [7] described it using lists of vertices and facets. Belforte et al [2] recently presented a synthesis of the technique. All the same, the main stages of the calculation are as follows. One starts with an orthotope shaped domain of investigation $\mathcal{D}_q$ aligned with the coordinate axes: $\{\lambda\in\mathbb{R}^{+}/\lambda_i\leq d_i,\ \ \forall i\in\{1,\ldots,q\}\}$. Taking successively the different inequalities of (7) into account, the domain is then truncated to the polytope domain of the admissible values of λ.

Notice that we have all along assumed that the value of θ_c is known. One can obtain the most precise θ_c estimator using a simplex type of algorithm [9]. One retains the value of θ_c that minimizes either criterion (4a) or criterion (4b).

To complete this study, we will now propose a method for representing the abstract space $\mathcal{A}(\theta_k)$ in the parametric domain and for an a posteriori verification that the computed values are really the right ones. We will begin by presenting a cartography of the parameter space for different values of δ_ε. We will then move on to show how to verify that the abstract space $\mathcal{A}(\theta_k)$ really respect the constraints.

5. Map of the parameter space

Leaving aside the development of θ_k, the constraint in (1) can simply be written as follows:

$$\forall k\in\{1,\cdots,h\},\ \tilde{y}_k\in\mathcal{A}\left(\tilde{x}_k{}^{T}\theta_k+\varepsilon_k\right) \qquad (8)$$

with $\mathcal{A}(\theta_k)=\Theta$ and $\mathcal{A}(\varepsilon_k)=\left[-\delta_\varepsilon,\delta_\varepsilon\right]$

For this constraint to be verified, it suffices that, for all k in $\{1, \ldots, h\}$, the following conditions are satisfied:

$$\exists\theta\in\Theta\,/\,\tilde{y}_k\leq\sup_{\varepsilon_k\in[-\delta_\varepsilon,\delta_\varepsilon]}\left(\tilde{x}_k{}^{T}\theta+\varepsilon_k\right)$$

$$\exists\theta'\in\Theta\,/\,\tilde{y}_k\geq\inf_{\varepsilon_k\in[-\delta_\varepsilon,\delta_\varepsilon]}\left(\tilde{x}_k{}^{T}\theta'+\varepsilon_k\right)$$

Taking into account the fact that $\mathcal{A}(\varepsilon_k)=[-\delta_\varepsilon,\delta_\varepsilon]$, one obtains:

$$\exists\theta\in\Theta\cap\Theta_k^{+}\text{ with }\Theta_k^{+}=\left\{\theta\,/\,\tilde{y}_k-\delta_\varepsilon\leq\tilde{x}_k{}^{T}\theta\right\} \qquad (9a)$$

$$\exists\theta'\in\Theta\cap\Theta_k^{-}\text{ with }\Theta_k^{-}=\left\{\theta'\,/\,\tilde{y}_k+\delta_\varepsilon\geq\tilde{x}_k{}^{T}\theta'\right\} \qquad (9b)$$

Θ_k^{+} represents the set of the parameters θ for which the values of $\tilde{x}_k{}^{T}\theta+\delta_\varepsilon$ are greater than $\tilde{y}_k$ while Θ_k^{-} represents the set for which the values of $\tilde{x}_k{}^{T}\theta'-\delta_\varepsilon$ are smaller than $\tilde{y}_k$. Consequently, the intersections, $\Theta\cap\Theta_k^{+}$ and $\Theta\cap\Theta_k^{-}$ represent the elements of the domain Θ which ensure that, at the instant k, the measurement $\tilde{y}_k$ is inside the boundaries of the abstract space $\mathcal{A}\left(\tilde{x}_k{}^{T}\theta_k+\varepsilon_k\right)$. It follows that the intersection $\Theta\cap\Theta_1^{+}$ contains the parameters θ of the domain Θ for which the values of $\tilde{x}_1{}^{T}\theta+\delta_\varepsilon$ are bigger than $\tilde{y}_1$. At the instant k=2, it is the intersection $\Theta\cap\Theta_2^{+}$ that contains the parameters θ for which the derived values of $\tilde{x}_2{}^{T}\theta+\delta_\varepsilon$ are bigger than $\tilde{y}_2$. The subset Θ for which the values of $\tilde{x}_1{}^{T}\theta+\delta_\varepsilon$ and $\tilde{x}_2{}^{T}\theta+\delta_\varepsilon$ are respectively bigger than $\tilde{y}_1$ and $\tilde{y}_2$, corresponds necessarily to the intersections $\Theta\cap\Theta_1^{+}\cap\Theta_2^{+}$. Proceeding in this way, step by step, one deduces that the parameters of $\Theta\cap\Theta_1^{+}\cap\cdots\cap\Theta_h^{+}$ lead to values of θ such that $\tilde{x}_k{}^{T}\theta+\delta_\varepsilon\leq\tilde{y}_k,\ \forall k\in\{1,\cdots,h\}$ (cf. figure II).

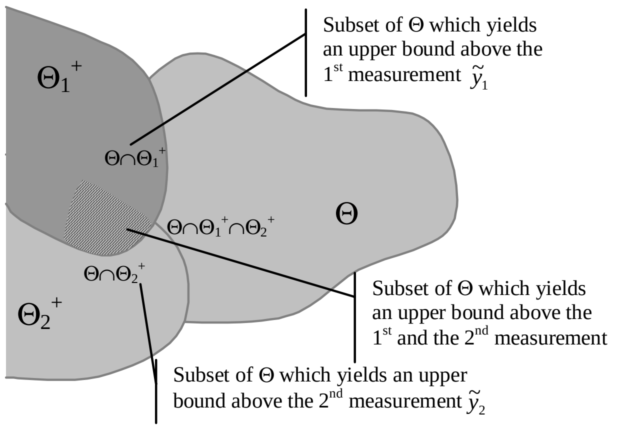

figure II – Elementary supremal and the parameter set domains.

Reasoning in analogous manner, it can be deduced that the Θ domain will explain the set of measurements, provided there exists at least one value of Θ belonging to $\Theta_1^{+}\cap\cdots\cap\Theta_h^{+}$ and at least one belonging to $\Theta_1^{-}\cap\cdots\cap\Theta_h^{-}$. In other words, letting $\Theta^{+}=\bigcap_{k=1}^{h}\Theta_k^{+}$ and $\Theta^{-}=\bigcap_{k=1}^{h}\Theta_k^{-}$, we can conclude that the following two constraints must be verified:

$$\Theta\cap\Theta^{+}\neq\{\varnothing\}\text{ and }\Theta\cap\Theta^{-}\neq\{\varnothing\} \qquad (10)$$

We have just shown that relation (9a) implies relation (10). To establish that both relations are equivalent, it remains to show that (10) also implies (9a). We observe that relations (10) imply that:

$$\exists\theta \in \left(\Theta \cap \Theta_1^+ \cap \cdots \cap \Theta_h^+\right)$$

and $$\exists\theta' \in \left(\Theta \cap \Theta_1^- \cap \cdots \cap \Theta_h^-\right)$$

Referring to the definitions of the elementary domains Θ_k^+ and Θ_k^-, one can easily show that there exists a $(\theta,\theta')\in\Theta^2$ / $\forall k\in\{1,\dots,h\}$:

$$\tilde{x}_k^{\mathrm{T}}\theta' - \delta_\varepsilon \le \tilde{y}_k \le \tilde{x}_k^{\mathrm{T}}\theta + \delta_\varepsilon$$

Consequently, relation (10) is necessarily verified. Since both implications are verified, the equivalence between (9a) and (10) is established.

We will refer to the two domains Θ^+ and Θ^- respectively as supremum and infimum bounding domains. These domains can be constructed independently of the domain Θ; each of both domains is just a function of the measurements and δ_ε. The evaluation of the domains $\Theta^+(\delta_\varepsilon)$ and $\Theta^-(\delta_\varepsilon)$ for different values of δ_ε provides the maps of the parametric space.

We will only give a brief description of the method used in computing these maps. In fact, notwithstanding the slight differences which we will bring out in the sequel, the technique employed is quite similar to the one employed to construct the polytope of admissible values λ.

Consider the bounding domain $\Theta^+(\delta_\varepsilon)$. A priori, such a domain is not bounded. It is therefore indispensable to conduct its investigation on a bounded domain $\mathcal{D}$ of $\mathbb{R}^p$. We will refer to $\mathcal{D}$ as the domain of investigation. This domain will advantageously be chosen in the form of an orthotope, which is aligned with the coordinate axes. In decomposing $x=[x_1,\dots,x_p]^{\mathrm{T}}$, we will write $x\in\mathcal{D}\Leftrightarrow\{x/ \forall i\in\{1,\dots,p\},\ d_i^-\le x_i\le d_i^+\}$ where d_i^+ and d_i^- are real numbers. The problem is then to evaluate the domain $\mathcal{D}\cap\Theta^+$ or $\left(\left(\left(\mathcal{D}\cap\Theta_1^+\right)\cap\Theta_2^+\right)\cdots\cap\Theta_h^+\right)$.

To show that $\mathcal{D}\cap\Theta^+$ is a polytope, we begin by showing that it is a polyhedron and then proceed to verify that it is convex.

The intersection of the orthotope $\mathcal{D}$ with the half-space of parameters $\Theta_1^+ = \{\theta / \tilde{y}_1 - \delta_\varepsilon \le \tilde{x}_1^{\mathrm{T}}\theta\}$, determined by the hyperplane $\Lambda_1^+ = \{\theta / \tilde{y}_1 - \delta_\varepsilon = \tilde{x}_1^{\mathrm{T}}\theta\}$, is necessarily a polyhedron. We deduce that $\mathcal{D}\cap\Theta^+$ is a polyhedron. Generally speaking, the intersection of a polyhedron of the parameter space with a domain $\Theta_k^+ = \{\theta / \tilde{y}_k - \delta_\varepsilon \le \tilde{x}_k^{\mathrm{T}}\theta\}$, determined by the hyperplane $\Lambda_k^+ = \{\theta / \tilde{y}_k - \delta_\varepsilon = \tilde{x}_k^{\mathrm{T}}\theta\}$, is a polyhedron. The domain $\mathcal{D}$ and the bounding domains Θ_k^+ are all convex. Their intersection is therefore convex. It follows that $\mathcal{D}\cap\Theta^+$ is a polytope. One of its interesting properties is that it is perfectly defined by its vertices. The technique for generating such polytopes was previously mentioned. It allows the computation of the polytope's set of the characteristic vertices.

Considering different values for δ_ε, with

$$0\le\delta_\varepsilon\le \inf_{\theta_c\in\mathbb{R}^p}\Big(\sup_{k\in\{1,\cdots,h\}}\big(|\tilde{y}_k - \tilde{x}_k^{\mathrm{T}}\theta_c|\big)\Big)$$

we obtained a family of maps, $\mathcal{D}\cap\Theta^+(\delta_\varepsilon)$ and $\mathcal{D}\cap\Theta^-(\delta_\varepsilon)$, of the parameter space contained in the investigation domain $\mathcal{D}$. Each one of the maps is made up of two polytopes specified by their vertices. We will designate the vertex coordinate sets of $\mathcal{D}\cap\Theta^+(\delta_\varepsilon)$ and $\mathcal{D}\cap\Theta^-(\delta_\varepsilon)$ respectively by Σ^+ and Σ^-.

We recall that to explain all the observations, the abstract space $\Theta=\mathcal{A}(\theta_k)$, which corresponds to a θ_c-centered parallelotope of the parameter space, must verify relations (10). More precisely, Θ must contain at least one point in the domain $\Theta^+(\delta_\varepsilon)$ and at least one point in the domain $\Theta^-(\delta_\varepsilon)$. Notice the domain Θ is a function only of central invariant parameter vectors θ_c and the characteristic λ: $\Theta=\Theta(\theta_c, \lambda)$.

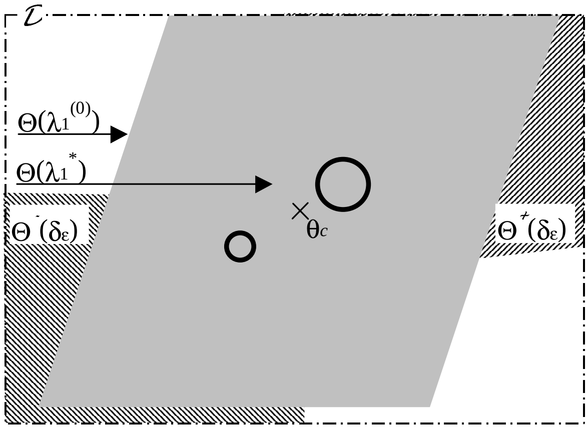

figure III – Searching for Θ: λ is a scalar quantity.

One easily sees the relevance of the solution $\mathcal{A}(\theta_k)$. Since $\Theta^+(\delta_\varepsilon)$ and $\Theta^-(\delta_\varepsilon)$ are both convex and Θ is a parallelotope, it is easy to verify condition (10). Indeed, it suffices to ensure either that there exists at least a vertex which verifies all the inequalities of the elementary domains Θ_k^+ of (9a) or that one of the vertices of $\Theta_k^+(\delta_\varepsilon)$ belongs to the domain Θ. The latter can also be described by means of inequalities [11]. It must similarly be verified that Θ and Θ^- do in fact intersect. Where dimensions are small, the results can assessed graphically (cf. figure III).

We presented in [11] how to solve the problem of the characterization defined in (1) and (4a) using the cartography of the parameter space. The solution however becomes more complex when the shape of Θ is indeterminate.

6. Example

By way of illustration, let us start by considering the simple example of a linear model of dimension $p=2$ for which the available measurements $\tilde{y}_k$ and $\tilde{x}_k$ ($k\in\{1,\ldots,3\}$) are:

$$\begin{bmatrix}\tilde{y}_1\\ \tilde{y}_2\\ \tilde{y}_3\end{bmatrix}=\begin{bmatrix}1.5\\ 0.8\\ 2.4\end{bmatrix},\ \begin{bmatrix}\tilde{x}_1^{\mathrm{T}}\\ \tilde{x}_2^{\mathrm{T}}\\ \tilde{x}_3^{\mathrm{T}}\end{bmatrix}=\begin{bmatrix}1.1 & 0.1\\ 0.2 & 1.2\\ 1.3 & 0.8\end{bmatrix}.$$

We first assume an equal distribution of the uncertainties of the two parameters of θ_k. In other words, λ is a scalar quantity and T_0 the two-order identity matrix. In the parameter space, the target abstract domain Θ is therefore a square with θ_c as center. The domains $\mathcal{D}\cap\Theta^+(\delta_\varepsilon)$ and $\mathcal{D}\cap\Theta^-(\delta_\varepsilon)$ in figure IV were drawn using three distinct values of the error δ_ε. Using the additive errors δ_ε, we calculated two best estimators successively using the Euclidean norm θ_c=[1.45,0,49]$^{\mathrm{T}}$ and the infinite norm θ_c=[1.43,0.53]$^{\mathrm{T}}$. We next evaluated an estimator which both verifies (1) and minimizes (4a). This was done according to the two-stage method presented in section 4 above. The associated estimator, baptized the most precise estimator (MP1), corresponds to the central value θ_c=[1.40,0.52]$^{\mathrm{T}}$. Figure IV shows that the bigger the equation error δ_ε, the closer the two domains Θ^+ and Θ^-; the domain Θ becomes smaller and the parameter uncertainties in turn are less significant.

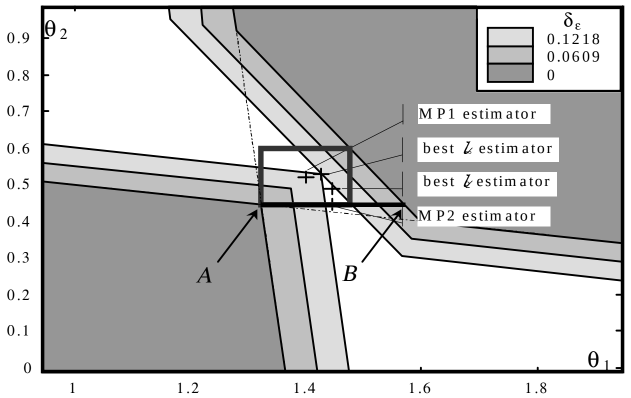

Figure IV – Mapping of parameter space

The square drawn in the center of the figure corresponds to the domain Θ associated with the estimator MP1 and the optimal pair (δ_ε=0,λ=7.69.10^{-2}) in the sense of the precision criterion (4a). It is also possible that the uncertainties associated with the two parameters differ. In this case, there are two parameters λ_1 and λ_2, and the matrix $T(\lambda)$ is as described in (3b) with t_1=[1,0]$^{\mathrm{T}}$ and t_2=[0,1]$^{\mathrm{T}}$. We denoted the corresponding estimator by MP2. The optimal domain Θ (figure IV) is a straight line segment centered on θ_c=[1.44,0.45]$^{\mathrm{T}}$. Its calculation is based on (δ_ε=0, λ_1=0.12, λ_2=0).

The upper chart of figure V presents the plot of the scalar parameter λ against the additive error δ_ε for different estimators, other than MP2. The latter cannot be represented in this plane. As can be seen, when the additive error becomes too small to embrace all the model errors $\tilde{y}_k-\tilde{x}_k\theta_c$, the algorithm has the effect of increasing the parameter uncertainties. On the one hand, when the additive error δ_ε exceeds $\sup_k\left(\left|\tilde{y}_k-\tilde{x}_k\theta_c\right|\right)$, λ becomes zero: the additive error δ_ε then explains all the observations.

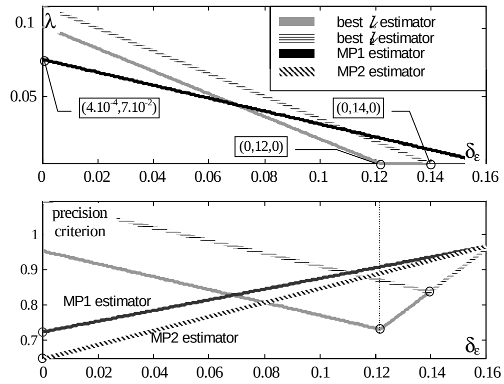

Figure V – Results of parameters characterization

The lower chart of the figure also presents the plot of the precision criterion of (4). As the curve is made up of straight line segments and the precision criteria linearly dependent on λ, it is natural to expect piece-wise linear criteria. For λ=0, the criterion simplifies to $2h\delta_\varepsilon$ which explains the fact that the curves $J(\delta_\varepsilon,\lambda(\delta_\varepsilon))$ for the different estimator θ_c coincide. We also added the criterion corresponding to the estimator MP2. λ is in this case a two-component vector. This choice naturally yields the most precise abstract domain.

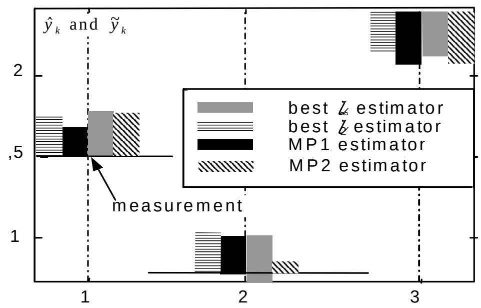

Figure VI – Corresponding uncertainties on estimations

The figure VI presents the abstract spaces $\mathcal{A}\left(\tilde{x}_k{}^T\theta_c+\tilde{x}_k{}^T T(\lambda)\upsilon_k+\varepsilon_k\right)$ corresponding to the different choices of estimators. For each of the estimators, the characteristics (δ_ε,λ) minimize the associated precision criterion.

7. Conclusion

We have proposed an algorithm for characterizing uncertainties in static linear models. In addition to the determination of the best distribution of uncertainties with respect to the precision, we also showed how to evaluate the central parametric values corresponding to the greatest precision. This was accomplished through an additional stage of minimization based on simplex method. Contrary to traditional techniques, the resulting central estimator fully exploits available knowledge about the distribution of uncertainties. The proposed technique may be extended to dynamical systems provided that they are not represented by a recursive form. However, in this case, if parameter uncertainties remains easy to represent, uncertainties on measurements are difficult to take into account.

8. Bibliography

[1] Barmish, B.R., and Sankaran, J., "The propagation of parametric uncertainty via polytopes," *IEEE transaction on automatic control*, vol. AC-24, n°2, 1979, pp.346-349.

[2] Belforte, G., Bona, B., and Cerone, V., "Parameter estimation algorithms for a set membership description of uncertainty," *Automatica*, vol.26, n°5, 1990, pp.887-898. Also in [5], Belforte, G., and Tay, T. T., "Recursive estimation algorithm for linear models with set membership error," pp.83-99.

[3] Fogel, E., and Huang, Y. F., "On the value of information in system identification-bounded noise case," *Automatica*, vol.18, n°2, 1982, pp.229-238.

[4] Jaulin, L., and Walter, E., "Set inversion via interval analysis for non linear bounded-error estimation," *Automatica*, vol. 29, n°4, 1993, pp.1053-1064.

[5] Milanese, M., and Belforte, G., "Estimation theory and uncertainty intervals evaluation in presence of unknown but bounded errors: linear families of models," *IEEE transaction on automatic control*, vol. AC-27, n°2, 1982, pp.408-414.

[6] Milanese, M., Norton, J., Piet-Lahanier, H., and Walter, E., (Ed.), "Bounding approaches to system identification," Plenum Press, New-York & London, 1996.

[7] Mo, S. H., and Norton, J. P., "Recursive parameter-bounding algorithms which compute polytope bounds," *2nd IMACS proceedings, 12th World Congress on scientific computation*, Paris, 1988, pp.127-132.

[8] Moore, R. E., "Methods and applications of interval analysis," SIAM, Philadelphia, Pennsylvania.

[9] Nelder, J. A., and Maid, R., 1965, "A simplex method for function minimization," *Computer Journal*, vol.7, 1979, pp.308-313.

[10] Neumaier, A., "Interval methods for systems of equations," Cambridge University Press, Cambridge, 1990.

[11] Ploix, S., "Diagnostic des systèmes incertains : l’approche bornante," Ph.D. of the Institut National Polytechnique de Lorraine, 1998.

[12] Walter, E., and Piet-Lahanier, H., "Exact and recursive description of feasible parameter set for bounded error models," *26th IEEE Conference on Decision and Control*, Los Angeles, pp.1921-1922, 1987.